1 Relations

1.1 Binary relations

A (binary) *relation* R from set U to set V is a subset of the Cartesian product $U{\times}V$. If $(u,v){\in}R$, we say that u is in relation R to v. We usually denote this by $u\,R\,v$. Set U is called the *domain* of the relation and V its *range* (or: *codomain*). If $U{=}V$ we call R an *(endo)relation on* U.

1.1 Examples.

(a) "Is the mother of" is a relation from the set of all women to the set of all people. It consists of all pairs (*person1* , *person2*) where *person1* is the mother of *person2* . Of course, this relation also is an (endo)relation on the set of people.

(b) "There is a train connection between" is a relation on the set of cities in the Netherlands.

(c) The equality relation " $=$ " is a relation on every set. This relation is often denoted by I (and also called the "identity" relation). Because, however, every set has its "own" identity relation we sometimes use subscription to distinguish all these different identity relations. That is, for every set U we define I_U by:

$$I_U \ = \ \{ \ (u,u) \mid u{\in}U \ \} \quad .$$

Whenever no confusion is possible and it is clear which set is intended, we drop the subscript and write just I instead of I_U, and in ordinary mathematical language we use " $=$ ", as always. So, for any set U and for all $u,v{\in}U$, we have: $u\,I\,v \ \Leftrightarrow \ u{=}v$.

(d) Integer n divides integer m, notation $n|m$, if there is an integer $q{\in}\mathbb{Z}$ such that $q{*}n{=}m$. Divides $|$ is the relation on $\mathbb{Z}$ that consists of all pairs $(n,m) \in \mathbb{Z} \times \mathbb{Z}$ with $(\exists q : q{\in}\mathbb{Z} : q{*}n{=}m)$.

(e) "Less than" ($<$) and "greater than" ($>$) are relations on $\mathbb{R}$, and on $\mathbb{Q}, \mathbb{Z}$, and $\mathbb{N}$ as well, and so are "at most" ($\leq$) and "at least" ($\geq$).

(f) The set $\{ (a,p) \, , \, (b,p) \, , \, (b,q) \, , \, (c,q) \}$ is a relation from $\{ a,b,c \}$ to $\{ p,q \}$.

(g) The set $\{ (x,y) \in \mathbb{R}^2 \mid y = x^2 \}$ is a relation on $\mathbb{R}$.

(h) Let Ω be a set, then "is a subset of" ($\subseteq$) is a relation on the set of all subsets of Ω .

Besides binary relations we can also consider n-ary relations for any $n{\geq}0$. An n-ary relation on sets $U_0, \cdots, U_{n-1}$ is a subset of the Cartesian product $U_0 \times \cdots \times U_{n-1}$. Unless stated otherwise, in this text relations are binary.

Let R be a relation from set U to set V. Then for each element $u{\in}U$ we define $[\,u\,]_R$ as a subset of V, as follows:

$$[u]_R = \{v \in V \mid u R v\} \quad .$$

(Sometimes $[u]_R$ is also denoted by $R(u)$.) This set is called the (R-)image of u. Similarly, for $v \in V$ a subset of U called ${}_R[v]$ is defined by:

$${}_R[v] = \{u \in U \mid u R v\} \quad ,$$

which is called the (R-)pre-image of v.

1.2 Definition. If R is a relation from *finite* set U to *finite* set V, then R can be *represented* by means of a so-called *adjacency matrix*; sometimes this is convenient because it allows computations with (finite) relations to be carried out in terms of matrix calculations. We will see examples of this later.

With m for the *size* −the number of elements− of U and with n for the size of V, sets U and V can be represented by finite sequences, by numbering their elements. That is, we assume $U = \{u_1, \cdots, u_m\}$ and we assume $V = \{v_1, \cdots, v_n\}$. The adjacency matrix of relation R then is an $m \times n$ matrix A_R, say, the elements of which are 0 or 1 only, and defined by, for all $i,j : 1 \le i \le m \wedge 1 \le j \le n$:

$$A_R[i,j] = 1 \quad \Leftrightarrow \quad u_i \, R \, v_j \quad .$$

Here $A_R[i,j]$ denotes the element of matrix A_R at row i and column j. Note that this definition is equivalent to stating that $A_R[i,j] = 0$ if and only if $\neg(u_i \, R \, v_j)$, for all i,j. Actually, adjacency matrices are *boolean* matrices in which, for the sake of conciseness, true is encoded as 1 and false as 0; thus, we might as well state that: $A_R[i,j] \Leftrightarrow u_i \, R \, v_j$.
□

Notice that this representation is not *unique*: the elements of finite sets can be assigned numbers in very many ways, and the distribution of 0's and 1's over the matrix depends crucially on how the elements of the two sets are numbered. For instance, if U has m elements it can be represented by $m!$ different sequences of length m; thus, a relation between sets of sizes m and n admits as many as $m! * n!$ (potentially different) adjacency matrices for its representation. Not surprisingly, if $U = V$ it is *good practice* to use one and the same element numbering for the two U's (in $U \times U$). If $1 \le i \le m$ then the set $[u_i]_R$ is represented by the row with index i in the adjacency matrix, that is:

$$[u_i]_R \;=\; \{\, v_j \mid 1 \le j \le n \wedge A_R[i,j] = 1 \,\} \quad .$$

Similarly, for $1 \le j \le n$ we have:

$${}_R[v_j] \;=\; \{\, u_i \mid 1 \le i \le m \wedge A_R[i,j] = 1 \,\} \quad .$$

1.3 Examples.

(a) An adjacency matrix for the relation $\{(a,p), (b,p), (b,q), (c,q)\}$ from $\{a,b,c\}$ to $\{p,q\}$ is:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

(b) Another adjacency matrix for the same relation and the same sets is obtained by reversing the order of the elements in one set: if we take (c, b, a) instead of (a, b, c) and if we keep (p, q) (as above), then the adjacency matrix becomes:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} .$$

Note that standard set notation is *over specific*, as the order of the elements in an expression like $\{a, b, c\}$ is irrelevant: $\{a, b, c\}$ and $\{c, b, a\}$ are *the same* set! Therefore, when we decide to represent a relation by an adjacency matrix we need not take the order of the set's elements for granted: we really have quite some freedom here.

(c) An adjacency matrix for the identity relation on a set of size n is the $n \times n$ *identity matrix* I_n:

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} .$$

This matrix is unique, that is, independent of how the elements of the set are ordered, provided we stick to the convention of good practice, that both occurrences of the same set are ordered in the same way.

(d) An adjacency matrix of relation $\leq$ on the set $\{1, 2, 3, 4, 5\}$ is the upper triangular matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

* * *

Some relations have special properties, which deserve to be named.

1.4 Definition. Let R be a relation on a set U. Then R is called:

- *reflexive*, if for all $x \in U$ we have: $x R x$;
- *irreflexive*, if for all $x \in U$ we have: $\neg(x R x)$;
- *symmetric*, if for all $x, y \in U$ we have: $x R y \;\Leftrightarrow\; y R x$;
- *antisymmetric*, if for all $x, y \in U$ we have: $x R y \wedge y R x \;\Rightarrow\; x = y$;
- *transitive*, if for all $x, y, z \in U$ we have: $x R y \wedge y R z \;\Rightarrow\; x R z$.

1.5 Examples. We consider some of the examples given earlier:

(a) "Is the mother of" is a relation on the set of all people. It is irreflexive, antisymmetric, and not transitive.

(b) "There is a train connection between" is symmetric and transitive. If one is willing to accept traveling over a zero distance as a train connection, then this relation also is reflexive.

(c) On every set relation "equals" ($=$) is reflexive, symmetric, and transitive.

(d) Relation "divides" ($|$) is reflexive, antisymmetric, and transitive.

(e) "Less than" ($<$) and "greater than" ($>$) on $\mathbb{R}$ are irreflexive, antisymmetric, and transitive, whereas "at most" ($\leq$) and "at least" ($\geq$) are reflexive, antisymmetric, and transitive.

(f) The relation $\{ (x, y) \in \mathbb{R}^2 \mid y = x^2 \}$ is neither reflexive nor irreflexive.

□

For any relation R the proposition $(\forall x, y : x, y \in U : x R y \Leftrightarrow y R x)$ is (logically) equivalent to the proposition $(\forall x, y : x, y \in U : x R y \Rightarrow y R x)$, which is (formally) weaker. Hence, relation R is symmetric if $x R y \Rightarrow y R x$, for all $x, y \in U$. To *prove* that R is symmetric, therefore, it suffices to prove the latter, weaker, version of the proposition, whereas to *use* (in other proofs) that R is symmetric we may use the stronger version.

1.6 Lemma. Every reflexive relation R on set U satisfies: $u \in [u]_R$, for all $u \in U$.

Proof. By calculation:

$$
\begin{array}{ll}
 & u \in [u]_R \\
\Leftrightarrow & \quad \{ \text{ definition of } [u]_R \ \} \\
 & u \, R \, u \\
\Leftrightarrow & \quad \{ \ R \text{ is reflexive } \} \\
 & \mathsf{true}
\end{array}
$$

□

1.7 Lemma. Every symmetric relation R on set U satisfies: $v \in [u]_R \Leftrightarrow u \in [v]_R$, for all $u, v \in U$.

Proof. By calculation:

$$
\begin{array}{ll}
& v \in [u]_R \\
\Leftrightarrow & \quad \{ \text{definition of } [u]_R \} \\
& u\,R\,v \\
\Leftrightarrow & \quad \{ R \text{ is symmetric} \} \\
& v\,R\,u \\
\Leftrightarrow & \quad \{ \text{definition of } [v]_R \} \\
& u \in [v]_R
\end{array}
$$

□

* * *

If R is a relation on a finite set S, then special properties like reflexivity, symmetry and transitivity can be read off from the adjacency matrix. For example, R is reflexive if and only if the main diagonal of R's adjacency matrix contains 1's only, that is if $A_R[i,i] = 1$ for all (relevant) i.

Relation R is symmetric if and only if the transposed matrix A_R^{T} equals A_R. The *transposed matrix* M^{T} of an $m \times n$ matrix M is the $n \times m$ matrix defined by, for all i, j:

$$M^{\mathrm{T}}[j,i] \;=\; M[i,j] \quad .$$

1.2 Equivalence relations

Relations that are reflexive, symmetric, and transitive deserve some special attention: they are called *equivalence relations*.

1.8 Definition. A relation R is an *equivalence relation* if and only if it is reflexive, symmetric, and transitive.
□

If elements u and v are related by an equivalence relation R, that is, if uRv, then u and v are also called "equivalent (under R)".

1.9 Example. On every set relation "equals" $(=)$ is an equivalence relation.

1.10 Example. Consider the plane $\mathbb{R}^2$ and in it the set S of straight lines. We call two lines in S parallel if and only if they are equal or do not intersect. Hence, two lines in S are parallel if and only if their slopes are equal. Being parallel is an equivalence relation on the set S.

1.11 Example. We consider a fixed $d \in \mathbb{Z}$, $d > 0$, and we define a relation R on $\mathbb{Z}$ by: $m\,R\,n$ if and only if $m - n$ is divisible by d. The latter can be formulated as $(m - n)\,\mathsf{mod}\,d = 0$, and a more traditional mathematical rendering of this is $m = n\,(\mathsf{mod}\,d)$. Thus defined, R is an equivalence relation.
$\square$

Actually, the last two examples are instances of Theorem 1.13. Before giving this theorem, first we introduce equivalence classes and prove a lemma about them.
If R is an equivalence relation on set U, then, for every $u \in U$ the set $[u]_R$ is called the *equivalence class of* u. Because equivalence relations are reflexive we have, as we have seen in lemma 1.6: $u \in [u]_R$, for all $u \in U$. From this it follows immediately that equivalence classes are nonempty. Equivalence classes have several other interesting properties. For example, the equivalence classes of two elements are equal if and only if these elements are equivalent:

1.12 Lemma. Every equivalence relation R on set U satisfies, for all $u, v \in U$:

$$[u]_R = [v]_R \;\Leftrightarrow\; u\,R\,v \quad .$$

Proof. The left-hand side of this equivalence contains the function $[\cdot]_R$, whereas the right-hand side does not. To eliminate $[\cdot]_R$ we rewrite the left-hand side first:

$$\begin{array}{ll} & [u]_R = [v]_R \\ \Leftrightarrow & \quad \{ \text{ set equality } \} \\ & (\forall x : x \in U : x \in [u]_R \Leftrightarrow x \in [v]_R\,) \\ \Leftrightarrow & \quad \{ \text{ definition of } [\cdot]_R \text{ } \} \\ & (\forall x : x \in U : u\,R\,x \;\Leftrightarrow\; v\,R\,x\,) \quad , \end{array}$$

hence, the lemma is equivalent to:

$$(\forall x : x \in U : u\,R\,x \;\Leftrightarrow\; v\,R\,x\,) \;\Leftrightarrow\; u\,R\,v \quad .$$

This we prove by mutual implication.

$$\begin{array}{ll} \text{“}\Rightarrow\text{”:} & (\forall x : x \in U : u\,R\,x \;\Leftrightarrow\; v\,R\,x\,) \\ \Rightarrow & \quad \{ \text{ instantiation } x := v \text{ } \} \\ & u\,R\,v \Leftrightarrow v\,R\,v \\ \Leftrightarrow & \quad \{ \text{ } R \text{ is an equivalence relation, so it is reflexive } \} \\ & u\,R\,v \quad . \end{array}$$

“$\Leftarrow$”: Assuming $u\,R\,v$ and for any $x \in U$ we prove $u\,R\,x \Leftrightarrow v\,R\,x$, again by mutual implication:

$$\begin{array}{ll} & u\,R\,x \\ \Leftrightarrow & \quad \{ \text{ assumption } \} \end{array}$$

$$
\begin{array}{ll}
 & u\,R\,v \;\wedge\; u\,R\,x \\
\Leftrightarrow & \quad \{\ R \text{ is an equivalence relation, so it is symmetric }\} \\
 & v\,R\,u \;\wedge\; u\,R\,x \\
\Rightarrow & \quad \{\ R \text{ is an equivalence relation, so it is transitive }\} \\
 & v\,R\,x \quad ,
\end{array}
$$

and:

$$
\begin{array}{ll}
 & v\,R\,x \\
\Leftrightarrow & \quad \{\ \text{assumption }\} \\
 & u\,R\,v \;\wedge\; v\,R\,x \\
\Rightarrow & \quad \{\ R \text{ is an equivalence relation, so it is transitive }\} \\
 & u\,R\,x \quad ,
\end{array}
$$

which concludes the proof of this lemma.

□

1.13 Theorem. A relation R on a set U is an equivalence relation if and only if a set V and a function $f : U \to V$ exists such that

$$x\,R\,y \;\Leftrightarrow\; f(x) = f(y)$$

for all $x, y \in U$.

Proof.

First we prove the 'if'-part: assume such a V and f exists; we have to prove that R is an equivalence relation.

Choose $x \in U$ arbitrary, then xRx holds since $f(x) = f(x)$. So R is reflexive.

Choose $x, y \in U$ arbitrary for which then xRy holds. Then $f(x) = f(y)$, so also $f(y) = f(x)$, so yRx holds. So R is symmetric.

Choose $x, y, z \in U$ arbitrary for which xRy and yRz holds. Then $f(x) = f(y)$ and $f(y) = f(z)$, so also $f(x) = f(z)$. Hence xRz holds. This proves that R is transitive.

Combining these three properties we conclude that R is an equivalence relation, concluding the 'if'-part.

Next we prove the 'only if'-part: assume R is an equivalence relation; we have to find V and f having the required property.

Choose V to be the set of all subsets of U and define $f(x) = [\,x\,]_R$ for all $x \in U$. Then the required property

$$x\,R\,y \;\Leftrightarrow\; f(x) = f(y)$$

holds due to Lemma 1.12.

□

1.14 Example. We reconsider Example 1.11. The predicate $(m-n) \,\mathsf{mod}\, d = 0$ is equivalent to $m \,\mathsf{mod}\, d = n \,\mathsf{mod}\, d$, so with $\mathbb{Z}$ both for set U and for set V, function f, defined by $f(m) = m \,\mathsf{mod}\, d$, for all $m \in \mathbb{Z}$, does the job.
□

As a further investigation of equivalence classes we now observe that they are either disjoint or equal:

1.15 Lemma. Every equivalence relation R on set U satisfies, for all $u, v \in U$:

$$[u]_R \cap [v]_R = \emptyset \;\lor\; [u]_R = [v]_R \quad .$$

Proof. This proposition is equivalent to:

$$[u]_R \cap [v]_R \neq \emptyset \;\Rightarrow\; [u]_R = [v]_R \quad ,$$

which we prove as follows:

$$
\begin{array}{ll}
 & [u]_R \cap [v]_R \neq \emptyset \\
\Leftrightarrow & \quad \{ \text{ definition of } \emptyset \text{ and } \cap \text{ } \} \\
 & (\exists x : x \in U : x \in [u]_R \land x \in [v]_R \,) \\
\Leftrightarrow & \quad \{ \text{ definition of } [\,\cdot\,]_R \text{ } \} \\
 & (\exists x : x \in U : u\,R\,x \land v\,R\,x \,) \\
\Rightarrow & \quad \{ \text{ } R \text{ is symmetric and transitive } \} \\
 & (\exists x : x \in U : u\,R\,v \,) \\
\Rightarrow & \quad \{ \text{ predicate calculus } \} \\
 & u\,R\,v \\
\Leftrightarrow & \quad \{ \text{ lemma 1.12 } \} \\
 & [u]_R = [v]_R
\end{array}
$$
□

The equivalence classes of an equivalence relation "cover" the set:

1.16 Lemma. Every equivalence relation R on set U satisfies: $(\bigcup_{u:\, u \in U} [u]_R) = U$.

Proof. By mutual set inclusion. On the one hand, every equivalence class is a subset of U, that is: $[u]_R \subseteq U$, for all $u \in U$; hence, their union, $(\bigcup_{u:\, u \in U} [u]_R)$, is a subset of U as well. On the other hand, we have for every $v \in U$ that $v \in [v]_R$, so, also $v \in (\bigcup_{u:\, u \in U} [u]_R)$. Hence, U is a subset of $(\bigcup_{u:\, u \in U} [u]_R)$ too.
□

These lemmas show that the equivalence classes of an equivalence relation form a, so-called, *partition* of set U.

1.17 Definition. A partition of set U is a set Π of nonempty and disjoint subsets of U, the union of which equals U. Formally, that set Π is a partition of U means the conjunction of:

(a) $(\forall X : X \in \Pi : X \subseteq U \wedge X \neq \emptyset)$

(b) $(\forall X, Y : X, Y \in \Pi : X \cap Y = \emptyset \vee X = Y)$

(c) $(\bigcup_{X : X \in \Pi} X) = U$

□

Clause (a) in this definition expresses that the elements of a partition of U are nonempty subsets of U, clause (b) expresses that the sets in a partition are disjoint, whereas clause (c) expresses that the sets in a partition together "cover the whole" U. Phrased differently, clause (b) and (c) together express that every element of U is an element of *exactly one* of the sets in the partition.

Conversely, every partition also represents an equivalence relation. Every element of set U is element of exactly one of the subsets in the partition. "Being in the same subset" (in the partition) is an equivalence relation.

1.18 Theorem. Every partition Π of a set U represents an equivalence relation on U, the equivalence classes of which are the sets in Π.

Proof. Because Π is a partition, every element of U is an element of a unique subset in Π. Now, the relation "being elements of the same subset in Π" is an equivalence relation. Formally, we prove this by defining a function $\varphi : U \to \Pi$, as follows, for all $u \in U$ and $X \in \Pi$:

$$\varphi(u) = X \quad \Leftrightarrow \quad u \in X \quad .$$

Thus defined, φ is a function indeed, because for every $u \in U$ one and only one $X \in \Pi$ exists satisfying $u \in X$. Now relation $\sim$ on U, defined by, for all $u, v \in U$:

$$u \sim v \quad \Leftrightarrow \quad \varphi(u) = \varphi(v) \quad ,$$

is an equivalence relation −Theorem 1.13!−. Furthermore, by its very construction φ satisfies $u \in \varphi(u)$ and, hence, $\varphi(u)$ is the equivalence class of u, for all $u \in U$.

□

1.3 Operations on Relations

Relations between two sets are subsets of the Cartesian Product of these two sets. Hence, all usual set operations can be applied to relations as well. In addition, relations admit of some dedicated operations that happen to have nice algebraic properties. It is even possible to develop a viable Relational Calculus, but this falls outside the scope of this text.

These relational operations play an important role in the mathematical study of programming constructs, such as recursion and data structures. They are also useful in some theorems about graphs. We will see applications of this later.

1.3.1 Set operations

- For sets U and V, the *extreme relations* from U to V are the *empty relation* $\emptyset$ and the *full relation* $U\times V$. For the sake of brevity and symmetry, we denote these two relations by $\perp$ ("bottom") and $\top$ ("top"), respectively; element wise, they satisfy, for all $u\in U$ and $v\in V$:

  $$\neg(u\perp v) \;\;\wedge\;\; u\top v \quad .$$

  For example, every relation R satisfies: $\perp\subseteq R$ and $R\subseteq\top$, which is why we call $\perp$ and $\top$ the extreme relations.

- If R and S are relations, with the same domain and and with the same range, then $R\cup S$, and $R\cap S$, and $R\setminus S$ are relations too, between the same sets as R and S, and with the obvious meaning. The complement R^{C} of relation R is $\top\setminus R$.

- These operations have their usual algebraic properties. In particular, $\top$ and $\perp$ are the identity elements of $\cup$ and $\cap$, respectively: $R\cup\perp = R$ and $R\cap\top = R$. They are zero elements as well, that is: $R\cup\top = \top$ and $R\cap\perp = \perp$.

1.3.2 Transposition

With every relation R from set U to set V a corresponding relation exists from V to U that contains (v,u) if and only if $(u,v)\in R$. This relation is called the *transposition* of R and is denoted by R^{T}. (Some mathematicians use R^{-1}, but this may be confusing: transposition is not the same as inversion, especially with functions.) Formally, transposition is defined as follows.

1.19 Definition. For every relation R from set U to set V, relation R^{T} from V to U is defined by, for all $v\in V$ and $u\in U$:

$$v\,R^{\mathrm{T}}\,u \;\;\Leftrightarrow\;\; u\,R\,v \quad .$$

1.20 Lemma. Transposition distributes over all set operations, that is:

$$\begin{array}{l}
\perp^{\mathrm{T}} = \perp \;\text{ and: }\; \top^{\mathrm{T}} = \top \;\;; \\
(R\cup S)^{\mathrm{T}} \;=\; R^{\mathrm{T}}\cup S^{\mathrm{T}} \;\;; \\
(R\cap S)^{\mathrm{T}} \;=\; R^{\mathrm{T}}\cap S^{\mathrm{T}} \;\;; \\
(R\setminus S)^{\mathrm{T}} \;=\; R^{\mathrm{T}}\setminus S^{\mathrm{T}} \;\;; \\
(R^{\mathrm{C}})^{\mathrm{T}} = (R^{\mathrm{T}})^{\mathrm{C}} \quad .
\end{array}$$

1.21 Lemma. Transposition is its own *inverse*, that is, every relation R satisfies:

$$(R^{\mathrm{T}})^{\mathrm{T}} = R \quad .$$

□

For finite relations there is a direct connection between relation transposition and matrix transposition:

1.22 Lemma. If A_R is an adjacency matrix for relation R then $(A_R)^{\mathrm{T}}$ is an adjacency matrix for R^{T}.

1.23 Examples. Properties of relations, like (ir)reflexivity and (anti)symmetry, can now be expressed concisely by means of relational operations; for R a relation on set U:

- "R is reflexive" $\Leftrightarrow$ $I_U \subseteq R$
- "R is irreflexive" $\Leftrightarrow$ $I_U \cap R = \bot$
- "R is symmetric" $\Leftrightarrow$ $R^{\mathrm{T}} = R$
- "R is antisymmetric" $\Leftrightarrow$ $R \cap R^{\mathrm{T}} \subseteq I_U$

$\square$

Unfortunately, transitivity cannot be expressed so nicely in terms of the set operations. For this we need yet another operation on relations, which turns out to be quite useful for other purposes too.

1.3.3 Composition

Let R be a relation from U to V and let S be a relation from V to W. If $u\,R\,v$, for some $v \in V$ and if $v\,S\,w$, for that *same* v, then we say that u is related to w in the *composition* of R and S, written as $R\,;S$. So, the composition of R and S is a relation from U to W. Phrased differently, in this composition $u \in U$ is related to $w \in W$ if u and w are "connected via" some "intermediate" value in V. This is rendered formally as follows.

1.24 Definition. If R is a relation from U to V, and if S is a relation from V to W, then the composition $R\,;S$ is the relation from U to W defined by, for all $u \in U$ and $w \in W$:

$$u\,(R\,;S)\,w \;\Leftrightarrow\; (\exists v : v \in V : u R v \wedge v S w) \quad .$$

1.25 Example. Let $R = \{(1,2),(2,3),(2,4),(3,1),(3,3)\}$ be a relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ and let $S = \{(1,a),(2,c),(3,a),(3,d),(4,b)\}$ be a relation from $\{1,2,3,4\}$ to $\{a,b,c,d\}$. Then the composition $R\,;S$ is the relation $\{(1,c),(2,a),(2,b),(2,d),(3,a),(3,d)\}$, from $\{1,2,3\}$ to $\{a,b,c,d\}$.

1.26 Lemma. For any endorelation R we have:

$$R \text{ is transitive} \;\Leftrightarrow\; (R\,;R) \subseteq R \quad .$$

Proof. Assume R is transitive. Let $(x, y) \in R; R$. Then there exists z such that $(x, z) \in R$ and $(z, y) \in R$. By transitivity we conclude that $(x, y) \in R$. So we have proved $R; R \subseteq R$.

Conversely, assume $R; R \subseteq R$. Let xRy and yRz. Then by definition of composition we have $(x, z) \in R; R$. Since $R; R \subseteq R$ we conclude $(x, z) \in R$, by which we have proved that R is transitive.
□

1.27 Lemma. The identity relation is the identity of relation composition. More precisely, every relation R from set U to set V satisfies: $I_U \,;R = R$ and $R\,; I_V = R$.
Proof. We prove the first claim; the second is similar.

If $(x, y) \in I_U; R$ then there exists $z \in U$ such that $(x, z) \in I_U$ and $(z, y) \in R$. From the definition of I_U we conclude $x = z$, so from $(z, y) \in R$ we conclude $(x, y) \in R$.

Conversely, let $(x, y) \in R$. Then $(x, x) \in I_U$, so $(x, y) \in I_U; R$.

□

1.28 Lemma. Relation composition is associative, that is, all relations R, S, T satisfy: $(R\,;S)\,;T = R\,;(S\,;T)$.

Proof. For all u, x we calculate:

$$
\begin{array}{ll}
 & u\,((R\,;S)\,;T)\,x \\
\Leftrightarrow & \quad \{ \text{ definition of } ; \} \\
 & (\exists w :: u\,(R;S)\,w \wedge wTx\,) \\
\Leftrightarrow & \quad \{ \text{ definition of } ; \} \\
 & (\exists w :: (\exists v :: uRv \wedge vSw) \wedge wTx\,) \\
\Leftrightarrow & \quad \{ \wedge \text{ over } \exists \} \\
 & (\exists w :: (\exists v :: uRv \wedge vSw \wedge wTx\,)\,) \\
\Leftrightarrow & \quad \{ \text{ swapping dummies } \} \\
 & (\exists v :: (\exists w :: uRv \wedge vSw \wedge wTx\,)\,) \\
\Leftrightarrow & \quad \{ \text{ (almost) the same steps as above, in reverse order } \} \\
 & u\,(R\,;(S\,;T))\,x
\end{array}
$$

□

Remark: In other mathematical texts relation composition is sometimes called "(relational) product", denoted by infix operator $*$. From a formal point of view, this is harmless, of course, but it is important to keep in mind that composition is *not commutative*: generally, $R\,;S$ differs from $S\,;R$. This is the reason why we prefer to use an asymmetric symbol, "$;$", to denote composition: from a practical point of view the term "product" and the symbol "$*$" may be misleading.
□

An important property is that relation composition *distributes over arbitrary unions* of relations, both from the left and from the right:

1.29 Theorem. Every relation R and every collection Ω of relations satisfies:

$$R\ ;\ (\bigcup_{X : X \in \Omega} X) \ = \ (\bigcup_{X : X \in \Omega} R\,;X) \ \ ,$$

and also:

$$(\bigcup_{X : X \in \Omega} X)\ ;\ R \ = \ (\bigcup_{X : X \in \Omega} X\,;R) \ \ .$$

Proof. We prove the first claim; the second is similar.

$(x, y) \in R; (\bigcup_{X : X \in \Omega} X)$

$\quad\Leftrightarrow \qquad$ { definition composition }

$\exists z : (x, z) \in R \wedge (z, y) \in \bigcup_{X : X \in \Omega} X$

$\quad\Leftrightarrow \qquad$ { definition $\bigcup$ }

$\exists z : (x, z) \in R \wedge \exists X \in \Omega : (z, y) \in X$

$\quad\Leftrightarrow \qquad$ { property $\exists$ }

$\exists X \in \Omega : \exists z : (x, z) \in R \wedge (z, y) \in X$

$\quad\Leftrightarrow \qquad$ { definition composition }

$\exists X \in \Omega : (x, y) \in R; X$

$\quad\Leftrightarrow \qquad$ { definition $\bigcup$ }

$(x, y) \in \bigcup_{X : X \in \Omega} R; X$.

□

Corollary: Relation composition is *monotonic*, that is, for all relations R, S, T :

$$S \subseteq T \ \Rightarrow \ R\,;S \subseteq R\,;T \ \ \text{, and also:}$$

$$R \subseteq S \ \Rightarrow \ R\,;T \subseteq S\,;T \ \ .$$

□

* * *

The n-fold composition of a relation R with itself also is written as R^n , as follows.

1.30 Definition. (exponentiation of relations) For any (endo)relation R and for all natural n , we define (recursively):

$$R^0 = I \ \wedge \ R^{n+1} = R\,;R^n \ \ .$$

□

For example, the formula expressing transitivity of R, as in Lemma 1.26, can now also be written as: $R^2 \subseteq R$.

1.31 Lemma. For endorelation R and for all natural m and n:

$$R^{m+n} \;=\; R^m \,;R^n \quad .$$

Proof. We apply induction on m. For $m = 0$ using Lemma 1.27 we obtain

$$R^{0+n} \;=\; R^n \;=\; I; R^n \;=\; R^0; R^n.$$

For the induction step we assume the induction hypothesis $R^{m+n} = R^m; R^n$.

$$\begin{array}{rcll} R^{(m+1)+n} & = & R^{(m+n)+1} & \\ & = & R; R^{m+n} & \text{(definition)} \\ & = & R; (R^m; R^n) & \text{(induction hypothesis)} \\ & = & (R; R^m); R^n & \text{(associativity, Lemma 1.28)} \\ & = & R^{m+1}; R^n, & \text{(definition)} \end{array}$$

concluding the proof.
□

* * *

In the representation of relations by adjacency matrices, relation composition is represented by matrix multiplication. That is, if A_R is an adjacency matrix for relation R and if A_S is an adjacency matrix for relation S then the product matrix $A_R \times A_S$ is an adjacency matrix for the composition $R\,;S$. This matrix product is well-defined only if the number of columns of matrix A_R equals the number of rows of matrix A_S. This is true because the number of columns of A_R equals the size of the range of relation R. As this range also is the domain of relation S −otherwise composition of R and S is impossible− this size also equals the number of rows of A_S.

Recall that adjacency matrices actually are boolean matrices; hence, the matrix multiplication must be performed with boolean operations, not integer operations, in such a way that addition and multiplication boil down to disjunction ("or") and conjunction ("and") respectively. So, a formula like $(\Sigma j :: A_R[i,j] * A_S[j,k]\,)$ actually becomes: $(\exists j :: A_R[i,j] \wedge A_S[j,k]\,)$.

1.32 Example. Let $R = \{(1,2)\,,(2,3)\,,(2,4)\,,(3,1)\,,(3,3)\}$ be a relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ and let $S = \{(1,a)\,,(2,c)\,,(3,a)\,,(3,d)\,,(4,b)\}$ be a relation from $\{1,2,3,4\}$ to $\{a,b,c,d\}$. Then adjacency matrices for R and S are:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \text{, and: } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad .$$

The product of these matrices is an adjacency matrix for $R\,;S$:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad .$$

1.3.4 Closures

Some (endo)relations have properties, like reflexivity, symmetry, or transitivity, whereas other relations do not. For any such property, the *closure* of a relation with respect to that property is the *smallest extension* of the relation that does have the property. More precisely, it is fully characterized by the following definition.

1.33 Definition. (closure) Let $\mathcal{P}$ be a predicate on relations, then the $\mathcal{P}$-closure of relation R is the relation S satisfying the following three requirements:

(a) $R \subseteq S$,

(b) $\mathcal{P}(S)$,

(c) $R \subseteq X \wedge \mathcal{P}(X) \Rightarrow S \subseteq X$, for all relations X.

Indeed, (a) expresses that S is an *extension* of R, and (b) expresses that S has property $\mathcal{P}$, and (c) expresses that S is contained in every relation X that is an extension of R and that has property $\mathcal{P}$; this is what we mean by the *smallest* extension of R.

For instance, if a relation R already has property $\mathcal{P}$, so $\mathcal{P}(R)$ holds, then $S = R$ satisfies the properties (a), (b) and (c), so we conclude that then the $\mathcal{P}$-closure of R is R itself.

For any given property $\mathcal{P}$ and relation R the $\mathcal{P}$-closure of R need not exist, but if it exists it is unique, as is stated in the following theorem.

1.34 Theorem. If both S and S' satisfy properties (a), (b) and (c) from Definition 1.33, then $S = S'$.

Proof. By (a) and (b) for S we conclude $R \subseteq S$ and $\mathcal{P}(S)$, so by property (c) for S' we conclude $S' \subseteq S$.

By (a) and (b) for S' we conclude $R \subseteq S'$ and $\mathcal{P}(S')$, so by property (c) for S we conclude $S \subseteq S'$.

Combining $S' \subseteq S$ and $S \subseteq S'$ yields $S = S'$. □

remark: In this subsection we are studying properties of the general shape $\varphi(X) \subseteq X$, where φ is a *monotonic* function from relations to relations, and where parameter X is a relation. In a later chapter, on Partial Orders, we will see that for monotonic functions φ every relation does indeed have a closure with respect to that property. The requirements for such a closure now are that it is the smallest of all relations X satisfying:

$$R \subseteq X \wedge \varphi(X) \subseteq X \quad ,$$

which can be rewritten into this (logically equivalent) form:

$$R \cup \varphi(X) \subseteq X \quad .$$

The smallest relation having this property is the *intersection* of all relations having that property, that is: $(\bigcap_{X:\varphi(X)\subseteq X} X)$. We will also see that, under some additional conditions, this smallest relation is equal to the union of all "approximations from below", that is: $(\bigcup_{i:0\leq i} \varphi^i(\perp))$. As these are rather general properties, which are not specific to closures of relations, we will not elaborate this here.

□

* * *

The simplest possible property of relations is reflexivity. The reflexive closure of an (endo)relation R now is the smallest extension of R that is reflexive.

1.35 Theorem. The reflexive closure of a relation R is $R \cup I$.

Proof. We have to prove (a), (b) and (c) for $\mathcal{P}$ being reflexivity. Indeed, $R \subseteq R \cup I$, proving (a), and $R \cup I$ is reflexive since $I \subseteq R \cup I$, proving (b). For proving (c) assume that $R \subseteq X$ and X reflexive; we have to prove that $R \cup I \subseteq X$. Let $(x, y) \in R \cup I$. Then $(x, y) \in R$ or $(x, y) \in I$. If $(x, y) \in R$ then from $R \subseteq X$ we conclude $(x, y) \in X$; if $(x, y) \in I$ then from reflexivity we conclude that $(x, y) \in X$. So in both cases we have $(x, y) \in X$, so $R \cup I \subseteq X$, concluding the proof. □

1.36 Theorem. The symmetric closure of a relation R is $R \cup R^{\mathrm{T}}$.

Proof. We have to prove (a), (b) and (c) for $\mathcal{P}$ being symmetry. Indeed, $R \subseteq R \cup R^{\mathrm{T}}$, proving (a).

For proving (b) let $(x, y) \in R \cup R^{\mathrm{T}}$. If $(x, y) \in R$ then $(y, x) \in R^{\mathrm{T}} \subseteq R \cup R^{\mathrm{T}}$. If $(x, y) \in R^{\mathrm{T}}$ then $(y, x) \in (R^{\mathrm{T}})^{\mathrm{T}} = R \subseteq R \cup R^{\mathrm{T}}$. So in both cases $(y, x) \in R \cup R^{\mathrm{T}}$, proving that i$R \cup R^{\mathrm{T}}$ is symmetric, so proving (b).

For proving (c) assume that $R \subseteq X$ and X is symmetric; we have to prove that $R \cup R^{\mathrm{T}} \subseteq X$. Let $(x, y) \in R \cup R^{\mathrm{T}}$. If $(x, y) \in R$ then from $R \subseteq X$ we conclude $(x, y) \in X$. If $(x, y) \in R^{\mathrm{T}}$ then $(y, x) \in R \subseteq X$; since X is symmetric we conclude $(x, y) \in X$. So in both cases we have $(x, y) \in X$, concluding the proof. □

* * *

The game becomes more interesting when we ask for the *transitive closure* of a relation R.

We define

$$R^{+} \;=\; \bigcup_{i=1}^{\infty} R^{i} \;=\; R \cup R^{2} \cup R^{3} \cup R^{4} \cup \cdots .$$

1.37 Theorem. The transitive closure of a relation R is R^+.

Proof. We have to prove (a), (b) and (c) for $\mathcal{P}$ being transitivity. Indeed, $R \subseteq R^+$, proving (a).

For proving (b) we have to prove that R^+ is transitive. So let $(x,y),(y,z) \in R^+$. Since $R+ = \bigcup_{i=1}^{\infty} R^i$ there are $i,j \geq 1$ such that $(x,y) \in R^i$ and $(y,z) \in R^j$. So $(x,z) \in R^i; R^j = R^{i+j}$ by Lemma 1.31. Since $R^{i+j} \subseteq \bigcup_{i=1}^{\infty} R^i = R^+$ we conclude $(x,z) \in R^+$, concluding the proof of (b).

For proving (c) assume that $R \subseteq X$ and X is transitive; we have to prove that $R^+ \subseteq X$. For doing so first we prove that $R^n \subseteq X$ for all $n \geq 1$, by induction on n. For $n = 1$ this is immediate from the assumption $R \subseteq X$. So next assume the induction hypothesis $R^n \subseteq X$ and we will prove $R^{n+1} \subseteq X$. Let $(x,y) \in R^{n+1} = R; R^n$, so there exists z such that $(x,z) \in R$ and $(z,y) \in R^n$. Since $R \subseteq X$ we conclude $(x,z) \in X$, and by the induction hypothesis we conclude $(z,y) \in X$. Since X is transitive we conclude $(x,y) \in X$. Hence $R^{n+1} \subseteq X$. By the principle of induction we have proved $R^n \subseteq X$ for all $n \geq 1$.

For (c) we had to prove $R^+ \subseteq X$. So let $(x,y) \in R^+ = \bigcup_{i=1}^{\infty} R^i$. Then there exists $n \geq 1$ such that $(x,y) \in R^n$. Since $R^n \subseteq X$ we conclude $(x,y) \in X$, concluding the proof. $\square$

In computer science the notation $+$ is often used for 'one or more times', while notation $*$ is often used for 'zero or more times'. Consistent with this convention we define

$$R^* = \bigcup_{i=0}^{\infty} R^i = I \cup R \cup R^2 \cup R^3 \cup R^4 \cup \cdots .$$

It can be shown that R^* is the reflexive-transitive closure of R, that is, the $\mathcal{P}$-closure for $\mathcal{P}$ being the conjunction of reflexivity and transitivity.

1.4 Exercises

1. Give an example of a relation that is:

   (a) both reflexive and irreflexive;

   (b) neither reflexive nor irreflexive;

   (c) both symmetric and antisymmetric;

   (d) neither symmetric nor antisymmetric.

2. For each of the following relations, investigate whether it is (ir)reflexive, (anti-) symmetric, and/or transitive:

   (a) $R = \{ (x,y) \in \mathbb{R}^2 \mid x+1 < y \}$

   (b) $S = \{ (x,y) \in \mathbb{R}^2 \mid x < y+1 \}$

   (c) $T = \{ (x,y) \in \mathbb{Z}^2 \mid x < y+1 \}$

3. Prove that each irreflexive and transitive relation is antisymmetric.

4. Which of the following relations on set U, with $U = \{1,2,3,4\}$, is reflexive, irreflexive, symmetric, antisymmetric, or transitive?

   (a) $\{(1,3),(2,4),(3,1),(4,2)\}$;

   (b) $\{(1,3),(2,4)\}$;

   (c) $\{(1,1),(2,2),(3,3),(4,4),(1,3),(2,4),(3,1),(4,2)\}$;

   (d) $\{(1,1),(2,2),(3,3),(4,4)\}$;

   (e) $\{(1,1),(2,2),(3,3),(4,4),(1,2),(2,3),(3,4),(4,3),(3,2),(2,1)\}$.

5. Construct for each of the relations in Exercise 4 the adjacency matrix.

6. Let R be a relation on a set U. Prove that, if $[u]_R \neq \emptyset$, for all $u \in U$, and if R is symmetric and transitive, then R is reflexive.

7. The natural numbers admit addition but not subtraction: if $a < b$ the difference $a - b$ is undefined, because it is not a natural number. To achieve a structure in which all differences are defined we need the "integer numbers". These can be constructed from the naturals in the following way, a process called "definition by abstraction".

   We consider the set V of all pairs of natural numbers, so $V = \mathbb{N} \times \mathbb{N}$. On V we define a relation $\sim$, as follows, for all $a, b, c, d \in \mathbb{N}$:

   $$(a,b) \sim (c,d) \quad \Leftrightarrow \quad a+d = c+b \quad .$$

   (a) Prove that $\sim$ is an equivalence relation.

   (b) Formulate in words what this equivalence relation expresses.

   (c) We investigate the equivalence classes of $\sim$. Obviously, there is a class containing the pair $(0,0)$. Prove that, in addition, every *other* class contains *exactly one* pair of the shape either $(a,0)$ or $(0,a)$, so not both in the same class, with $1 \leq a$.

   (d) We call the pairs $(0,0)$, $(a,0)$ and $(0,a)$, with $1 \leq a$, the "representants" of the equivalence classes. These classes can now be ordered in the following way, by means of their representants:

   $$\cdots,(0,2),(0,1),(0,0),(1,0),(2,0),(3,0),\cdots \quad .$$

   We call these classes "integer numbers"; a more usual notation of the representants is:

   $$\cdots,-2,-1,0,+1,+2,+3,\cdots \quad .$$

   Thus, we obtain the integer numbers indeed. To illustrate this: define, on the set of representants, two binary operators pls and min that correspond with the usual "addition" and "subtraction". Also define the "less than" relation on the set of representants.

8. Prove that every reflexive and transitive endorelation R satisfies: $R^2 = R$.
9. Construct for each relation, named R here, in Exercise 4 an adjacency matrix for R^2.
10. Suppose R and S are finite relations with adjacency matrices A and B, respectively. Define adjacency matrices, in terms of A and B, for the relations $R \cup S$, $R \cap S$, $R \backslash S$, and R^{C}.
11. Suppose R and S are endorelations. Prove or disprove:
    (a) If R and S are reflexive, then so is $R\,;S$.
    (b) If R and S are irreflexive, then so is $R\,;S$.
    (c) If R and S are symmetric, then so is $R\,;S$.
    (d) If R and S are antisymmetric, then so is $R\,;S$.
    (e) If R and S are transitive, then so is $R\,;S$.
    (f) If R and S are equivalence relations, then so is $R\,;S$.
12. Prove that every endorelation R satisfying $R \subseteq I$ satisfies:
    (a) R is symmetric.
    (b) R is antisymmetric.
    (c) R is transitive.
13. Let $\mathcal{D}$ be the set of differentiable functions $f : \mathbb{R} \to \mathbb{R}$. On $\mathcal{D}$ we define a relation $\sim$ as follows, for all $f, g \in \mathcal{D}$:

    $$f \sim g \;\Leftrightarrow\; \text{“function } f - g \text{ is constant”} \;.$$

    Prove that $\sim$ is an equivalence relation. How can relation $\sim$ be defined in the way of Theorem 1.13 ?
14. We consider the relation $\sim$ on $\mathbb{Z}$ defined by, for all $x, y \in \mathbb{Z}$:

    $$x \sim y \;\Leftrightarrow\; (\exists z \in \mathbb{Z} : x - y = 7z)$$

    Prove that $\sim$ is an equivalence relation. Describe the equivalence classes of $\sim$. In particular, establish how many equivalence classes $\sim$ has.
15. Let R and S be two equivalence relations on a finite set U satisfying $R \subseteq S$.
    (a) Prove that every equivalence class of R is a subset of an equivalence class of S.
    (b) Let n_R be the number of equivalence classes of R and let n_S be the number of equivalence classes of S. Prove that $n_R \geq n_S$.

16. On the set $U = \{1,2,3,4,5,6\}$ define the relation

$$R \;=\; \{(i,i) \mid i \in U\} \cup \{(1,2),(2,1),(2,3),(3,2),(1,3),(3,1),(4,6),(6,4)\}.$$

Show that R is an equivalence relation. Establish what are the equivalence classes of R, in particular, how many equivalence classes R has, and how many elements each of them has.

17. We consider a linear vector space V and a (fixed) subspace W of V On V we define a relation $\sim$ by, for all $x,y \in V$:

$$x \sim y \;\Leftrightarrow\; x - y \in W \quad .$$

Prove that $\sim$ is an equivalence relation. Describe the equivalence classes for the special case that $V = \mathbb{R}^2$ W is the straight line given by the equation $x_1 + x_2 = 0$. Also characterize, for this special case, the equivalence relation in the way of Theorem 1.13.

18. An adjacency matrix for a relation R is: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Investigate whether R is (ir)reflexive, (anti)symmetric, and/or transitive.

19. Prove that $(R\,;S)^{\mathrm{T}} \;=\; S^{\mathrm{T}}\,;R^{\mathrm{T}}$.

20. (a) Prove that, for all sets A,B,C: $A \subseteq C \wedge B \subseteq C \;\Leftrightarrow\; A \cup B \subseteq C$.
    (b) Prove that, for all sets A,B: $A \subseteq B \;\Leftrightarrow\; A \cup B \;=\; B$ and also: $A \subseteq B \;\Leftrightarrow\; A \cap B \;=\; A$.
    (c) Prove that relation composition distributes over union, that is: $R\,;(S \cup T) \;=\; (R\,;S) \cup (R\,;T)$ and: $(R \cup S)\,;T \;=\; (R\,;T) \cup (S\,;T)$.
    (d) Using the previous result(s), prove that $;$ is *monotonic*, that is: $S \subseteq T \;\Rightarrow\; R\,;S \subseteq R\,;T$ and also: $R \subseteq S \;\Rightarrow\; R\,;T \subseteq S\,;T$.

21. Prove that, indeed, $R \cup R^{\mathrm{T}}$ is the smallest solution of equation, with unknown X: $R \subseteq X \wedge X^{\mathrm{T}} \subseteq X$.

22. Prove that $R\,;\bot \;=\; \bot$, for every relation R.

23. Prove that $\top$ is a solution of each of the equations (with unknown X) in Subsection 1.3.4.

24. We consider a relation R from U to V for which it is given that it is a function. Prove that R is surjective if and only if $I_V \;=\; R^{\mathrm{T}}\,;R$.

25. Relation R, on $\mathbb{Z}$, is defined by $m\,R\,n \Leftrightarrow m+1 = n$, for all $m,n \in \mathbb{Z}$. What is relation R^+?

26. For some given set Ω, a function ϕ, mapping subsets of Ω to subsets of Ω, is called *monotonic* if $X \subseteq Y \Rightarrow \phi(X) \subseteq \phi(Y)$, for all $X,Y \subseteq \Omega$.

(a) We consider the equations: $X : \phi(X) \subseteq X$ and: $X : \phi(X) = X$, and we *assume* they have smallest solutions; so, proving the *existence* of these smallest solutions is not the subject of this exercise. Prove that, if ϕ is monotonic then the smallest solutions of these equations are equal.

(b) For each of the closures in Subsection 1.3.4, define a function ϕ such that the corresponding equation is equivalent to $\phi(X) \subseteq X$; for each case, prove that ϕ is monotonic. What is Ω in these cases?

27. Prove that $R^* = I \cup R^+$ and that $R^+ = R\,;R^*$.

28. Prove that for every endorelation R: "R is transitive" $\Leftrightarrow$ $R^+ = R$.

29. We consider two endorelations R and S satisfying $R\,;S \subseteq S\,;R^+$. Prove that: $R^+\,;S \subseteq S\,;R^+$.

30. Let R, S be relations on a set U satisfying $R \subseteq S$. Prove that $R;R \subseteq S;S$.

31. Give an example of relations R, S on a set U satisfying $R;R \subseteq S;S$, but not $R \subseteq S$.

32. Let R, S be relations on a set U satisfying $R \subseteq S$. Prove that $R^+ \subseteq S^+$.

33. Let R, S be two relations on a set U, of which R is transitive and S is reflexive. Prove that

$$(R;S;R)^2 \subseteq (R;S)^3.$$

34. Let R, S be two relations on a set U.

(a) Prove that $(R;S)^n;R = R;(S;R)^n$ for all $n \geq 0$.

(b) Prove that $(R;S)^*;R = R;(S;R)^*$.

35. (a) Let R be an endorelation and let S be a transitive relation. Prove that:

$$R \subseteq S \;\Rightarrow\; R^+ \subseteq S \quad .$$

(b) Apply this, by defining suitable relations R and S, to prove that every function f on $\mathbb{N}$ satisfies:

$$(\forall i : 0 \leq i < n : f_i = f_{i+1}) \;\Rightarrow\; f_0 = f_n \quad \text{, for all } n \in \mathbb{N} \quad .$$

36. We call a relation on a set *inductive* if it admits proofs by Mathematical Induction. Formally, a relation R on a set V is inductive if, for every predicate P on V:

$$(\forall v : v \in V : (\forall u : u R v : P(u)) \Rightarrow P(v)) \;\Rightarrow\; (\forall v : v \in V : P(v)) \quad .$$

Prove that, for every relation R:

"R is inductive" $\Rightarrow$ "R^+ is inductive" .

Hint: To prove the right-hand side of this implication one probably will introduce a predicate P. To apply the (assumed) left-hand side of the implication one may select any predicate desired, not necessarily P: use predicate Q defined by, for all $v \in V$: $Q(v) = (\forall u : u R^* v : P(u))$.

37. On the natural numbers a distinction is often made between (so-called) "weak" (or "step-by-step") induction and "strong" (or "course-of-values") induction. Weak induction is the property that, for every predicate P on $\mathbb{N}$:

$$P(0) \wedge (\forall n :: P(n) \Rightarrow P(n{+}1)) \;\Rightarrow\; (\forall n :: P(n)) \quad ,$$

whereas strong induction is the property that, for every predicate P on $\mathbb{N}$:

$$(\forall n :: (\forall m : m{<}n : P(m)) \Rightarrow P(n)) \;\Rightarrow\; (\forall n :: P(n)) \quad ,$$

Show that the proposition:

$$\text{“weak induction”} \;\Rightarrow\; \text{“strong induction”} \quad ,$$

is a special case of the proposition in the previous exercise.

38. Construct an example, as simple as possible, illustrating that relation composition is not *commutative*, which means that it is *not* true that: $R\,;S = S\,;R$, for all relations R, S.

39. Suppose that endorelation R satisfies $I \cap R^+ = \bot$. What does this mean?

40. We investigate some well-known relations on $\mathbb{R}$:

    (a) What is the reflexive closure of $<$?
    (b) What is the symmetric closure of $<$?
    (c) What is the symmetric closure of $\leq$?
    (d) What is the reflexive closure of $\neq$?

    Compute for each of the relations in Exercise 4 their reflexive, symmetric, and transitive closures.